

ON L_p -ESTIMATES OF SOME SINGULAR INTEGRALS RELATED TO JUMP PROCESSES

R. MIKULEVICIUS AND H. PRAGARAUSKAS

ABSTRACT. We estimate fractional Sobolev and Besov norms of some singular integrals arising in the model problem for the Zakai equation with discontinuous signal and observation.

1. INTRODUCTION

In a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration of σ -algebras $\mathbb{F} = (\mathcal{F}_t)$ satisfying the usual conditions, the following linear stochastic integro-differential parabolic equation of the fixed order $\alpha \in (0, 2]$ was considered in Hölder classes (see [6]):

$$(1.1) \quad \begin{cases} du(t, x) = (A^{(\alpha)}u(t, x) + f(t, x))dt + \int_U g(t, x, v)q(dt, dv) & \text{in } E_{0,T}, \\ u(0, x) = u_0(x) & \text{in } \mathbf{R}^d, \end{cases}$$

where $E_{0,T} = [0, T] \times \mathbf{R}^d$, f is an \mathbb{F} -adapted measurable real-valued function on \mathbf{R}^{d+1} ,

$$\begin{aligned} & A^{(\alpha)}u(t, x) \\ &= \int_{\mathbf{R}_0^d} [u(t, x+y) - u(t, x) - (\nabla u(t, x), y)\chi^{(\alpha)}(y)]m^{(\alpha)}(t, y)\frac{dy}{|y|^{d+\alpha}} \\ & \quad + (b(t), \nabla u(t, x))1_{\alpha=1} + \sum_{i,j=1}^d B^{ij}(t)\partial_{ij}^2 u(t, x)1_{\alpha=2}, \quad (t, x) \in \mathbf{R}^{d+1}, \end{aligned}$$

$\chi^{(\alpha)}(y) = 1_{\alpha>1} + 1_{|y|\leq 1}1_{\alpha=1}$, $m^{(\alpha)}(t, y)$ is a bounded measurable real-valued function homogeneous in y of order zero, $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$, $b(t) = (b^1(t), \dots, b^d(t))$ is a bounded measurable function and $B(t) = (B^{ij}(t))$ is a bounded symmetric non-negative definite measurable matrix-valued function;

$$q(dt, dv) = p(dt, dv) - \Pi(dv)dt$$

Date: March 14, 2012.

1991 Mathematics Subject Classification. 60H15.

Key words and phrases. L_p -estimates of singular integrals, SPDEs, Lévy processes, Zakai equation.

is a martingale measure on a measurable space $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})$ ($p(dt, dv)$ is a Poisson point measure on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})$ with the compensator $\Pi(dv)dt$) and g is an \mathbb{F} -adapted measurable real-valued function on $\mathbf{R}^{d+1} \times U$. It is the model problem for the Zakai equation (see [16]) arising in the nonlinear filtering problem with discontinuous observation (see [6]). Let us consider the following example.

Example 1. Assume that the signal process X_t in R^d is defined by

$$X_t = X_0 + \int_0^t b(X_s)ds + W_t^\alpha, t \in [0, T],$$

where $b(x) = (b^i(x))_{1 \leq i \leq d}, x \in R^d$, are measurable and bounded W_t^α is a d -dimensional α -stable ($\alpha \in (1, 2)$) Lévy process. Suppose

$$W_t^\alpha = \int_0^t \int v[p(ds, dv) - m\left(\frac{v}{|v|}\right) \frac{dv ds}{|v|^{d+\alpha}}],$$

where $m(\frac{v}{|v|})$ is a smooth bounded function (it characterizes the intensity of the jumps of W^α in the direction $\frac{v}{|v|}$) and $p(ds, dv)$ is a Poisson point measure on $[0, \infty) \times R_0^d$ with

$$\mathbf{E}p(ds, dv) = m\left(\frac{v}{|v|}\right) \frac{dv ds}{|v|^{d+\alpha}}.$$

Assume X_0 has a density function $u_0(x)$, and the observation Y_t is discontinuous, with jump intensity depending on the signal, such that

$$Y_t = \int_0^t \int_{|y|>1} y \hat{p}(ds, dy) + \int_0^t \int_{|y|\leq 1} y \hat{q}(ds, dy),$$

where $\hat{p}(ds, dy)$ is a point measure on $[0, \infty) \times R_0^d$ not having common jumps with W^α with a compensator $\rho(X_t, y)\pi(dy)$ and $\hat{q}(dt, dy) = \hat{p}(dt, dy) - \pi(dy)dt$. Assume $C_1 \geq \rho(x, y) \geq c_1 > 0, \pi(dy)$ is a measure on R_0^d such that

$$\int |y|^2 \wedge 1 \pi(dy) < \infty,$$

and $\int [\rho(x, y) - 1]^2 \pi(dy)$ is bounded. Then for every function φ such that $E[\varphi(X_t)^2] < \infty$, the optimal mean square estimate for $\varphi(X_t)$, $t \in [0, T]$, given the past of the observations $F_t^Y = \sigma(Y_s, s \leq t)$, is of the form

$$\hat{\varphi}_t = \mathbf{E}[\varphi(X_t) | \mathcal{F}_t^Y] = \frac{\tilde{\mathbf{E}}[\varphi(X_t) \zeta_t | \mathcal{F}_t^Y]}{\tilde{\mathbf{E}}[\zeta_t | \mathcal{F}_t^Y]},$$

where ζ_t is the solution of the linear equation

$$d\zeta_t = \zeta_{t-} \int [\rho(X_{t-}, y) - 1] \hat{q}(dt, dy)$$

and $d\tilde{P} = \zeta(T)^{-1} dP$. Under assumptions of differentiability, one can easily show that if $v(t, x)$ is an $F = (F_{t+}^Y)$ -adapted unnormalized filtering density function

$$(1.2) \quad \tilde{\mathbf{E}} [\varphi(X_t) \zeta_t | \mathcal{F}_t^Y] = \int v(t, x) \psi(x) dx,$$

then it is a solution of the Zakai equation

$$(1.3) \quad \begin{aligned} dv(t, x) &= v(t, x) \int [\rho(x, y) - 1] \hat{q}(dt, dy) + \left\{ -\partial_i(b^i(x)v(t, x)) \right. \\ &\quad \left. + \int_{\mathbf{R}_0^d} [v(t, x+y) - v(t, x) - (\nabla v(t, x), y)] m\left(\frac{-y}{|y|}\right) \frac{dy}{|y|^{d+\alpha}} \right\}, \\ v(0, x) &= u_0(x). \end{aligned}$$

Since $Y_t, t \geq 0$, and $X_t, t \geq 0$, are independent with respect to \tilde{P} , for $u(t, x) = v(t, x) - u_0(x)$ we have an equation whose model problem is of the type given by (1.1). Indeed, according to [2], for any infinitely differentiable function φ on \mathbf{R}^d with compact support, the conditional expectation $\pi_t(\varphi) = \tilde{E} [\varphi(X_t) \zeta_t | \mathcal{F}_t^Y]$ satisfies the equation

$$\begin{aligned} d\pi_t(\varphi) &= \int \pi_t(\varphi[\rho(\cdot, y) - 1]) \hat{q}(dt, dy) + \pi_t \left\{ (b, \nabla \varphi) \right. \\ &\quad \left. + \int_{\mathbf{R}_0^d} [\varphi(\cdot + y) - \varphi - (\nabla \varphi, y) \chi^{(\alpha)}(y)] m\left(t, \frac{y}{|y|}\right) \frac{dy}{|y|^{d+\alpha}} \right\} dt. \end{aligned}$$

Assuming (1.2) and integrating by parts, we obtain (1.3).

In terms of Fourier transform,

$$A^{(\alpha)}v(x) = \mathcal{F}^{-1} \left[\psi^{(\alpha)}(t, \xi) \mathcal{F}v(\xi) \right] (x),$$

with

$$\begin{aligned} \psi^{(\alpha)}(t, \xi) &= i(b(t), \xi) 1_{\alpha=1} - \sum_{i,j=1}^d B^{ij}(t) \xi_i \xi_j 1_{\alpha=2} \\ &\quad - C \int_{S^{d-1}} |(w, \xi)|^\alpha \left[1 - i \left(\tan \frac{\alpha\pi}{2} \operatorname{sgn}(w, \xi) 1_{\alpha \neq 1} \right. \right. \\ &\quad \left. \left. - \frac{2}{\pi} \operatorname{sgn}(w, \xi) \ln |(w, \xi)| 1_{\alpha=1} \right) \right] m^{(\alpha)}(t, w) dw, \end{aligned}$$

where $C = C(\alpha, d)$ is a positive constant, S^{d-1} is the unit sphere in \mathbf{R}^d and dw is the Lebesgue measure on it. It was shown in [6] that in

Hölder classes the solution of (1.1) can be represented as

$$(1.4) \quad u(t, x) = Rf(t, x) + \tilde{R}g(t, x) + T_t u_0(x),$$

where

$$(1.5) \quad \begin{aligned} Rf(t, x) &= \int_0^t G_{s,t} * f(s, x) ds, \\ \tilde{R}g(t, x) &= \int_0^t \int_U G_{s,t} * g(s, x, v) q(ds, dv), \\ T_t u_0(x) &= G_{0,t} * u_0(x), \end{aligned}$$

with

$$G_{s,t}(x) = \mathcal{F}^{-1} \left(\exp \left\{ \int_s^t \psi^{(\alpha)}(r, \xi) dr \right\} \right) x, \quad s \leq t, x \in \mathbf{R}^d,$$

and $*$ denoting the convolution with respect to the space variable $x \in \mathbf{R}^d$. According to [11], $G_{s,t}$ is the density function of an α -stable distribution, and $A^{(\alpha)}$ is the fractional Laplacian if $b = 0$ and $m^{(a)} = 1$.

In order to estimate the L_p -norm of the fractional derivative

$$\partial^\alpha u(t, x) = -\mathcal{F}^{-1} [|\xi|^\alpha \mathcal{F} u(t, \xi)]$$

of u in (1.4), we need the estimates for $\partial^\alpha Rf$, $\partial^\alpha \tilde{R}g$ and $\partial^\alpha T_t u_0$. It was derived in [7], that

$$|\partial^\alpha Rf|_{L_p} \leq C |f|_{L_p}.$$

According to Corollary 2 below (it provides two-sided estimates for the moments of a martingale),

$$\mathbf{E} |\partial^\alpha \tilde{R}g|_{L_p}^p \leq C [\mathbf{E} I_1 + \mathbf{E} I_2],$$

where

$$(1.6) I_1 = \int_0^T \int_{\mathbf{R}^d} \left\{ \int_0^t \int_U [\partial^\alpha G_{s,t} * g(s, x, v)]^2 \Pi(dv) ds \right\}^{p/2} dx dt,$$

$$(1.7) I_2 = \int_0^T \int_0^t \int_{\mathbf{R}^d} \int_U |\partial^\alpha G_{s,t} * g(s, x, v)|^p \Pi(dv) dx ds dt.$$

In this paper, we estimate the singular integrals of I_1 - and I_2 -types related to $\tilde{R}g(t, x)$ in (1.5) in Sobolev and Besov spaces. If $\alpha = 2$ and B is $d \times d$ -identity matrix, the estimate of I_1 -type was proved in [5]. This estimate for (1.6) was generalized in [4] for the case $m^{(\alpha)} = 1$, $b = 0$ (in this case $A^{(a)}$ is the fractional Laplacian). Our derivation of an estimate for (1.6) follows a slightly different idea communicated by N.V. Krylov. The problem cannot be reduced to a case with fractional Laplacian. In fact, $m^{(\alpha)}$ can be zero on a substantial set (see Remark 1). The operator $\tilde{R}g$ in Hölder-Zygmund classes was estimated in [6].

The results of this paper were applied in [9] to solve the model problem above in the fractional Sobolev spaces.

The paper consists of five sections. In Section 2, we introduce the notation and state the main results. In Section 3, we derive the two-sided p -moment estimates of discontinuous martingales that explain the need to consider (1.6) and (1.7). In the last two sections, we present the proofs of the main results.

2. NOTATION, FUNCTION SPACES AND MAIN RESULTS

2.1. Notation. The following notation will be used in the paper.

Let $\mathbf{N}_0 = \{0, 1, 2, \dots\}$, $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$. If $x, y \in \mathbf{R}^d$, we write

$$(x, y) = \sum_{i=1}^d x_i y_i, \quad |x| = \sqrt{(x, x)}.$$

We denote by $C_0^\infty(\mathbf{R}^d)$ the set of all infinitely differentiable functions on \mathbf{R}^d with compact support.

We denote the partial derivatives in x of a function $u(t, x)$ on \mathbf{R}^{d+1} by $\partial_i u = \partial u / \partial x_i$, $\partial_{ij}^2 u = \partial^2 u / \partial x_i \partial x_j$, etc.; $\partial u = \nabla u = (\partial_1 u, \dots, \partial_d u)$ denotes the gradient of u with respect to x ; for a multiindex $\gamma \in \mathbf{N}_0^d$ we denote

$$\partial_x^\gamma u(t, x) = \frac{\partial^{|\gamma|} u(t, x)}{\partial x_1^{\gamma_1} \dots \partial x_d^{\gamma_d}}.$$

For $\alpha \in (0, 2]$ and a function $u(t, x)$ on \mathbf{R}^{d+1} , we write

$$\partial^\alpha u(t, x) = -\mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}u(t, \xi)](x),$$

where

$$\mathcal{F}h(t, \xi) = \int_{\mathbf{R}^d} e^{-i(\xi, x)} h(t, x) dx, \quad \mathcal{F}^{-1}h(t, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i(\xi, x)} h(t, x) dx.$$

The letters $C = C(\cdot, \dots, \cdot)$ and $c = c(\cdot, \dots, \cdot)$ denote constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.

2.2. Function spaces. Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of smooth real-valued rapidly decreasing functions. Let V be a Banach space with a norm $|\cdot|_V$. The space of V -valued tempered distributions we denote by $\mathcal{S}'(\mathbf{R}^d, V)$ ($f \in \mathcal{S}'(\mathbf{R}^d, V)$ is a continuous V -valued linear functional on $\mathcal{S}(\mathbf{R}^d)$).

For a V -valued measurable function h on \mathbf{R}^d and $p \geq 1$ we denote

$$|h|_{V,p}^p = \int_{\mathbf{R}^d} |h(x)|_V^p dx.$$

Further, for a characterization of our function spaces we will use the following construction (see [1]). By Lemma 6.1.7 in [1], there exists a function $\phi \in C_0^\infty(\mathbf{R}^d)$ such that $\text{supp } \phi = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$, $\phi(\xi) > 0$ if $2^{-1} < |\xi| < 2$ and

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Define the functions $\varphi_k \in \mathcal{S}(\mathbf{R}^d)$, $k = 1, \dots$, by

$$\mathcal{F}\varphi_k(\xi) = \phi(2^{-k}\xi),$$

and $\varphi_0 \in \mathcal{S}(\mathbf{R}^d)$ by

$$\mathcal{F}\varphi_0(\xi) = 1 - \sum_{k \geq 1} \mathcal{F}\varphi_k(\xi).$$

Let $\beta \in \mathbf{R}$ and $p \geq 1$. We introduce the Besov space $B_{pp}^\beta = B_{pp}^\beta(\mathbf{R}^d, V)$ of generalized functions $f \in \mathcal{S}'(\mathbf{R}^d, V)$ with finite norm

$$|f|_{B_{pp}^\beta(\mathbf{R}^d, V)} = \left\{ \sum_{j=0}^{\infty} 2^{j\beta p} |\varphi_j * f|_{V,p}^p \right\}^{1/p},$$

the Sobolev space $H_p^\beta(\mathbf{R}^d, V)$ of $f \in \mathcal{S}'(\mathbf{R}^d, V)$ with finite norm

$$\begin{aligned} (2.1) \quad |f|_{H_p^\beta(\mathbf{R}^d, V)} &= |\mathcal{F}^{-1}((1 + |\xi|^2)^{\beta/2} \mathcal{F}f)|_{V,p} \\ &= |(I - \Delta)^{\beta/2} f|_{V,p}, \end{aligned}$$

where I is the identity map and Δ is the Laplacian in \mathbf{R}^d , and the space $\tilde{H}_p^\beta(\mathbf{R}^d, V)$ of $f \in \mathcal{S}'(\mathbf{R}^d, V)$ with finite norm

$$(2.2) \quad |f|_{\tilde{H}_p^\beta(\mathbf{R}^d, V)} = \left\{ \int_{\mathbf{R}^d} \left(\sum_{j=0}^{\infty} 2^{2\beta j} |\varphi_j * f(x)|_V^2 \right)^{p/2} dx \right\}^{1/p}.$$

Similarly we introduce the corresponding spaces of generalized functions on $E_{a,b} = [a, b] \times \mathbf{R}^d$ and $\tilde{E}_{a,b} = \{(s, t, x) \in \mathbf{R}^{d+2} : a \leq s \leq t \leq b, x \in \mathbf{R}^d\}$.

The spaces $B_{pp}^\beta(E_{a,b}, V)$, $H_p^\beta(E_{a,b}, V)$ and $\tilde{H}_p^\beta(E_{a,b}, V)$ consist of all measurable $\mathcal{S}'(\mathbf{R}^d, V)$ -valued functions on $[a, b]$ with finite corresponding norms:

$$(2.3) \quad \begin{aligned} |f|_{B_{pp}^\beta(E_{a,b}, V)} &= \left\{ \int_a^b |f(t, \cdot)|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p dt \right\}^{1/p}, \\ |f|_{H_p^\beta(E_{a,b}, V)} &= \left\{ \int_a^b |f(t, \cdot)|_{H_p^\beta(\mathbf{R}^d, V)}^p dt \right\}^{1/p} \end{aligned}$$

and

$$(2.4) \quad |f|_{\tilde{H}_p^\beta(E_{a,b}, V)} = \left\{ \int_a^b |f(t, \cdot)|_{\tilde{H}_p^\beta(\mathbf{R}^d, V)}^p dt \right\}^{1/p}.$$

The spaces $B_{pp}^\beta(\tilde{E}_{a,b}, V)$, $H_p^\beta(\tilde{E}_{a,b}, V)$ and $\tilde{H}_p^\beta(\tilde{E}_{a,b}, V)$ consist of all measurable $\mathcal{S}'(\mathbf{R}^d, V)$ -valued functions on $\{(s, t) : a \leq s \leq t \leq b\}$ with finite corresponding norms:

$$(2.5) \quad \begin{aligned} |f|_{B_{pp}^\beta(\tilde{E}_{a,b}, V)} &= \left\{ \int_a^b \int_a^t |f(s, t, \cdot)|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p ds dt \right\}^{1/p}, \\ |f|_{H_p^\beta(\tilde{E}_{a,b}, V)} &= \left\{ \int_a^b \int_a^t |f(s, t, \cdot)|_{H_p^\beta(\mathbf{R}^d, V)}^p ds dt \right\}^{1/p} \end{aligned}$$

and

$$(2.6) \quad |f|_{\tilde{H}_p^\beta(\tilde{E}_{a,b}, V)} = \left\{ \int_a^b \int_a^t |f(s, t, \cdot)|_{\tilde{H}_p^\beta(\mathbf{R}^d, V)}^p ds dt \right\}^{1/p}.$$

For the scalar functions the norms (2.1) and (2.2) are equivalent (see [15], p. 15). Therefore, the norms (2.3) and (2.4) as well as (2.5) and (2.6) are equivalent.

If V is a separable Hilbert space, we will also use the spaces $\bar{B}_{pp}^\beta(\tilde{E}_{a,b}, V)$ and $\bar{H}_p^\beta(\tilde{E}_{a,b}, V)$ consisting of measurable $\mathcal{S}'(\mathbf{R}^d, V)$ -valued functions on $\{(s, t) : a \leq s \leq t \leq b\}$ with finite norms

$$|f|_{\bar{B}_{pp}^\beta(\tilde{E}_{a,b}, V)} = \left\{ \sum_{j=0}^{\infty} 2^{j\beta p} \int_a^b \int_{\mathbf{R}^d} \left(\int_a^t |\varphi_j * f(s, t, x)|_V^2 ds \right)^{p/2} dx dt \right\}^{1/p}$$

and

$$|f|_{\bar{H}_p^\beta(\tilde{E}_{a,b}, V)} = \left\{ \int_a^b \int_{\mathbf{R}^d} \left(\int_a^t |\mathcal{F}^{-1}((1+|\xi|^2)^{\beta/2} \mathcal{F}f)(s, t, x)|_V^2 ds \right)^{p/2} dx dt \right\}^{1/p}.$$

2.3. Main results. Throughout the paper we assume that the functions $b = b(t)$, $B = B(t)$ and $m^{(\alpha)}(t, y) \geq 0$ are measurable, $m^{(2)} = 0$ and

$$\int_{S^{d-1}} w m^{(1)}(t, w) dw = 0, t \in \mathbf{R}.$$

Also, we will need the following assumptions.

A. (i) The function $m = m(t, y) \geq 0$ is 0-homogeneous and differentiable in y up to $d_0 = \left[\frac{d}{2}\right] + 1$;

(ii) There is a constant K such that for each $\alpha \in (0, 2)$ and $t \in \mathbf{R}$

$$|b(t)| + |B(t)| + \sup_{\substack{|\gamma| \leq d_0, \\ |\xi|=1}} |\partial_y^\gamma m^{(\alpha)}(t, y)| \leq K.$$

B. There is a constant $\mu > 0$ such that

$$\sup_{t, |\xi|=1} \operatorname{Re} \psi^{(\alpha)}(t, \xi) \leq -\mu.$$

Remark 1. The assumption **B** holds with certain $\mu > 0$ if, for example,

$$\begin{aligned} \inf_{t, |\xi|=1} (B(t)\xi, \xi) &> 0, \alpha = 2, \\ \inf_{t, w \in \Gamma} m^{(\alpha)}(t, w) &> 0, \alpha \in (0, 2), \end{aligned}$$

for a measurable subset $\Gamma \subseteq S^{d-1}$ of a positive Lebesgue measure.

Given a measurable $\mathcal{S}'(\mathbf{R}^d, V)$ -valued function g on \mathbf{R} , we consider a linear operator \mathcal{I} that assigns to it a $\mathcal{S}'(\mathbf{R}^d, V)$ -valued function on $\{(s, t) : s \leq t\}$:

$$\mathcal{I}g(s, t, x) = G_{s,t} * g(s, x), s \leq t, x \in \mathbf{R}^d.$$

The main results of the paper are the two propositions given below. Proposition 1 in the case $V = L_p(U, \mathcal{U}, \Pi)$ is related to the integral I_2 in (1.7) and Proposition 2 in the case $V = L_2(U, \mathcal{U}, \Pi)$ is related to the integral I_1 in (1.6).

Proposition 1. *Let Assumptions **A** and **B** hold, $p \geq 2, \beta \in \mathbf{R}, -\infty \leq a < b \leq \infty$. Then the operator $\mathcal{I} : B_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E_{a,b}, V) \rightarrow \tilde{H}_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)$ is bounded: there is a constant $C = C(\alpha, K, \mu, p, d)$ such that*

$$(2.7) \quad \|\mathcal{I}g\|_{\tilde{H}_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)} \leq C \|g\|_{B_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E_{a,b}, V)}, g \in B_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E_{a,b}, V).$$

Since for the scalar functions the norms (2.5) and (2.6) are equivalent, we have the following statement.

Corollary 1. *Let $V = L_p(U, \mathcal{U}, \Pi)$. Then Proposition 1 holds with $\tilde{H}_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)$ replaced by $H_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)$.*

Proof. Let $V = L_p(U, \mathcal{U}, \Pi)$. If $\mathcal{I}g \in H_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)$, then Π -a.e. $\mathcal{I}g(\cdot, \cdot, v) \in H_p^{\beta+\alpha}(\tilde{E}_{a,b}, \mathbf{R})$. Since the norms (2.3) and (2.4) are equivalent for the scalar functions, we have

$$\begin{aligned} |\mathcal{I}g|_{H_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)}^p &= \int_a^b \int_U \int_{\mathbf{R}^d} |(I - \Delta)^{(\beta+\alpha)/2} \mathcal{I}g(t, x, v)|^p \Pi(dv) dx dt \\ &\leq C \int_a^b \int_U \int_{\mathbf{R}^d} \left(\sum_{j=0}^{\infty} 2^{2(\beta+\alpha)j} |\varphi_j * \mathcal{I}g(t, x, v)|^2 \right)^{p/2} dx \Pi(dv) dt, \end{aligned}$$

and by Minkowski inequality

$$\begin{aligned} &\int_a^b \int_U \int_{\mathbf{R}^d} \left(\sum_{j=0}^{\infty} 2^{2(\beta+\alpha)j} |\varphi_j * \mathcal{I}g(t, x, v)|^2 \right)^{p/2} dx \Pi(dv) dt \\ &\leq C \int_a^b \int_{\mathbf{R}^d} \left(\sum_{j=0}^{\infty} 2^{2(\beta+\alpha)j} |\varphi_j * \mathcal{I}g(t, x, \cdot)|_V^2 \right)^{p/2} dx dt \\ &= C |\mathcal{I}g|_{\tilde{H}_p^{\beta+\alpha}(\tilde{E}_{a,b}, V)}^p \end{aligned}$$

and the statement follows by Proposition 1. \square

Proposition 2. *Let Assumptions **A** (with d_0 replaced by $d_0 + 1$) and **B** hold, $p \geq 2$, $\beta \in \mathbf{R}$, $-\infty \leq a < b \leq \infty$, and let V be a separable Hilbert space.*

Then there is a constant $C = C(\alpha, K, \mu, p, d)$ such that

$$|\partial^{\alpha/2} \mathcal{I}g|_{\tilde{H}_p^{\beta}(\tilde{E}_{a,b}, V)} \leq C |g|_{H_p^{\beta}(E_{a,b}, V)}, \quad g \in H_p^{\beta}(E_{a,b}, V)$$

and

$$|\partial^{\alpha/2} \mathcal{I}g|_{\tilde{B}_{pp}^{\beta}(\tilde{E}_{a,b}, V)} \leq C |g|_{B_{pp}^{\beta}(E_{a,b}, V)}, \quad g \in B_{pp}^{\beta}(E_{a,b}, V).$$

3. MOMENT ESTIMATES OF DISCONTINUOUS MARTINGALES

The following two-sided moment estimate for discontinuous martingales should be well known (see e.g. [10] for this type of estimate from above). For the sake of completeness we provide its proof. Let $p(dt, dv)$ be a σ -finite point measure on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})$ with a dual predictable projection measure $\pi(dt, dv)$ such that $\pi(\{t\} \times U) = 0$, $t \geq 0$, and let $\mathcal{R}(\mathbb{F})$ be the progressive σ -algebra on $[0, \infty) \times \Omega$ (see [3]). Denote by L_{loc}^2 the space of all $\mathcal{R}(\mathbb{F}) \otimes \mathcal{U}$ -measurable functions $g(t, v) = g(\omega, t, v)$ such that \mathbf{P} -a.s.

$$\int_0^t \int_U g(s, v)^2 \pi(ds, dv) < \infty$$

for all t .

Lemma 1. *Let $p \geq 2, g \in L_{loc}^2$ and*

$$Q_t = \int_0^t \int_U g(s, v) q(ds, dv), t \geq 0.$$

Then there are constants $C = C(p)$ and $c = c(p) > 0$ such that for any \mathbb{F} -stopping time $\tau \leq T$

$$\begin{aligned} & c\mathbf{E} \left[\int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds) + \left(\int_0^\tau \int_U g(s, v)^2 \pi(dv, ds) \right)^{p/2} \right] \\ (3.1) & \leq \mathbf{E} \left[\sup_{t \leq \tau} |Q_t|^p \right] \\ & \leq C\mathbf{E} \left[\int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds) + \left(\int_0^\tau \int_U g(s, v)^2 \pi(dv, ds) \right)^{p/2} \right] \end{aligned}$$

Proof. Let

$$A_t = \int_0^t \int_U g(s, v)^2 p(ds, dv), \quad L_t = \int_0^t \int_U g(s, v)^2 \pi(dv, ds), \quad t \geq 0.$$

By the Burkholder–Davis–Gundy inequality (see [3]), there are positive constants c_p and C_p such that for each \mathbb{F} -stopping time τ

$$c_p \mathbf{E}[A_\tau^{p/2}] \leq \mathbf{E} \left[\sup_{t \leq \tau} |Q_t|^p \right] \leq C_p \mathbf{E}[A_\tau^{p/2}].$$

Denoting $q = p/2 \geq 1$, we have

$$A_\tau^q = \sum_{s \leq \tau} [(A_{s-} + \Delta A_s)^q - A_{s-}^q] = \int_0^\tau \int_U [(A_{s-} + g(s, v)^2)^q - A_{s-}^q] p(ds, dv)$$

and

$$\mathbf{E}[A_\tau^q] = \mathbf{E} \int_0^\tau \int_U [(A_{s-} + g(s, v)^2)^q - A_{s-}^q] \pi(dv, ds).$$

Since there are two positive constants c, C such that for all non-negative numbers a, b

$$C(b^q + a^{q-1}b) \geq (a + b)^q - a^q \geq c(b^q + a^{q-1}b),$$

we have

$$\begin{aligned} (3.2) \quad & C\mathbf{E} \int_0^\tau \int_U [|g(s, v)|^p + A_{s-}^{q-1} g(s, v)^2] \pi(dv, ds) \geq \mathbf{E}[A_\tau^q] \\ & \geq c\mathbf{E} \int_0^\tau \int_U [|g(s, v)|^p + A_{s-}^{q-1} g(s, v)^2] \pi(dv, ds). \end{aligned}$$

Hence,

$$\begin{aligned} c\mathbf{E} \int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds) &\leq \mathbf{E}[A_\tau^q] \\ &\leq C\mathbf{E} \left\{ \int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds) + A_\tau^{q-1} L_\tau \right\}. \end{aligned}$$

On the other hand, for $q > 1$,

$$L_\tau^q = q \int_0^\tau L_s^{q-1} dL_s$$

and

$$\mathbf{E}[L_\tau^q] = q\mathbf{E} \int_0^\tau L_s^{q-1} dA_s \leq q\mathbf{E}[L_\tau^{q-1} A_\tau].$$

According to Young's inequality, for each $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned} A_\tau^{q-1} L_\tau &\leq \varepsilon A_\tau^q + C_\varepsilon L_\tau^q, \\ L_\tau^{q-1} A_\tau &\leq \varepsilon L_\tau^q + C_\varepsilon A_\tau^q. \end{aligned}$$

Therefore, there is a constant C such that

$$\begin{aligned} \mathbf{E}[L_\tau^q] &\leq C\mathbf{E}[A_\tau^q], \\ \mathbf{E}[A_\tau^q] &\leq C\mathbf{E} \left\{ \int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds) + L_\tau^q \right\}, \\ \mathbf{E}[A_\tau^q] &\geq \mathbf{E} \int_0^\tau \int_U |g(s, v)|^p \pi(dv, ds), \end{aligned}$$

and the statement follows. \square

Corollary 2. *Let $p \geq 2$, $g = g(s, x, v)$ be such that \mathbf{P} -a.s.*

$$\int_0^T \int_U \int_{\mathbf{R}^d} g(s, x, v)^2 \pi(dv, ds) dx < \infty,$$

and

$$Q(t, x) = \int_0^t \int_U g(s, x, v) q(ds, dv), 0 \leq t \leq T.$$

Then

$$\begin{aligned} \mathbf{E} \sup_{s \leq \tau} |Q(s, \cdot)|_p^p &\sim \mathbf{E} \left\{ \int_0^\tau \int_U |g(s, \cdot, v)|_p^p \pi(dv, ds) + \right. \\ &\quad \left. + \left[\int_0^\tau \int_U g(s, \cdot, v)^2 \pi(dv, ds) \right]^{1/2} \right\}^p \end{aligned}$$

and

$$\begin{aligned} \mathbf{E} \int_0^T |Q(s, \cdot)|_p^p ds &\sim \mathbf{E} \int_0^T \sup_{s \leq t} |Q(s, \cdot)|_p^p dt \\ &\sim \mathbf{E} \left\{ \int_0^T \int_0^t \int_U |g(s, \cdot, v)|_p^p \pi(dv, ds) dt + \right. \\ &\quad \left. + \int_0^T \left| \left[\int_0^t \int_U g(s, \cdot, v)^2 \pi(dv, ds) \right]^{1/2} \right|_p^p dt \right\}, \end{aligned}$$

where $|f|_p^p = \int |f(x)|^p dx$ and \sim denotes the equivalence of norms.

4. PROOF OF PROPOSITION 1

Let us introduce the functions

$$\begin{aligned} \tilde{\varphi}_0 &= \varphi_0 + \varphi_1, \\ \tilde{\varphi}_j &= \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j \geq 1, \end{aligned}$$

where $\varphi_j, j \geq 0$, are defined in Subsection 2.2. Let

$$h_{s,t}^j(x) = \mathcal{F}^{-1} \left\{ \exp \left\{ \int_s^t \psi^{(\alpha)}(r, \xi) dr \right\} \mathcal{F} \tilde{\varphi}_j(\xi) \right\}(x), \quad j \geq 0.$$

According to Lemma 12 in [6] or inequality (36) and Lemma 16 in [8], there are constants $C, c > 0$ such that for all $s \leq t, j \geq 1$,

$$\begin{aligned} (4.1) \quad \int |h_{s,t}^j(x)| dx &\leq C e^{-c2^{j\alpha}(t-s)} \sum_{k \leq d_0} [2^{j\alpha}(t-s)]^k, \\ \int |h_{s,t}^0(x)| dx &\leq C. \end{aligned}$$

For $g \in B_{pp}^{\alpha - \frac{\alpha}{p}}(E_{a,b}, V)$, we set

$$g_j(t, \cdot) = g(t, \cdot) * \varphi_j, \quad j \geq 0.$$

Obviously,

$$\varphi_j * \mathcal{I}g(s, t, \cdot) = \mathcal{I}g_j(s, t, \cdot), \quad j \geq 0.$$

Since $\varphi_j = \varphi_j * \tilde{\varphi}_j, j \geq 0$, we have

$$\mathcal{I}g_j(s, t, x) = h_{s,t}^j * g_j(s, x).$$

Therefore, by Minkowski's inequality,

$$\begin{aligned}
|\mathcal{I}g|_{\tilde{H}_p^\beta(\tilde{E}_{a,b},V)}^p &= \int_a^b \int_a^t \int \left(\sum_{j=0}^{\infty} 2^{2\beta j} |\varphi_j * \mathcal{I}g(s,t,x)|_V^2 \right)^{p/2} dx ds dt \\
&= \int_a^b \int_a^t \int \left(\sum_{j=0}^{\infty} 2^{2\beta j} |h_{s,t}^j * g_j(s,x)|_V^2 \right)^{p/2} dx ds dt \\
&\leq \int_a^b \int_a^t \left(\sum_{j=0}^{\infty} 2^{2\beta j} \left\{ \int |h_{s,t}^j * g_j(s,x)|_V^p dx \right\}^{2/p} \right)^{p/2} ds dt.
\end{aligned}$$

By (4.1),

$$\begin{aligned}
\left\{ \int |h_{s,t}^j * g_j(s,x)|_V^p dx \right\}^{1/p} &\leq \int |h_{s,t}^j(x)| dx |g_j(s,\cdot)|_{V,p} \\
&\leq C e^{-c2^{\alpha j}(t-s)} |g_j(s,\cdot)|_{V,p}, \quad j \geq 0.
\end{aligned}$$

So,

$$\begin{aligned}
|\mathcal{I}g|_{\tilde{H}_p^\beta(\tilde{E}_{a,b},V)}^p &\leq \int_a^b \int_a^t \left(\sum_{j=0}^{\infty} 2^{2\beta j} \left\{ \int |h_{s,t}^j * g_j(s,x)|_V^p dx \right\}^{2/p} \right)^{p/2} ds dt \\
(4.2) \quad &\leq C \int_a^b \int_a^t \left(\sum_{j=0}^{\infty} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,p}^2 \right)^{p/2} ds dt \\
&= C \int_a^b \int_s^b \left(\sum_{j=0}^{\infty} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,p}^2 \right)^{p/2} dt ds.
\end{aligned}$$

If $p = 2$, we have immediately

$$\begin{aligned}
|\mathcal{I}g|_{H_2^\beta(\tilde{E}_{a,b},V)}^2 &\leq C \int_a^b \int_s^b \sum_{j=0}^{\infty} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,2}^2 dt ds \\
&\leq C \int_a^b \sum_{j=0}^{\infty} 2^{2\beta j} 2^{-\alpha j} |g_j(s,\cdot)|_{V,2}^2 ds.
\end{aligned}$$

If $p > 2$, we split the sum in (4.2) as follows:

$$\begin{aligned}
\sum_{j=0}^{\infty} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,p}^2 &= \sum_{j \in J} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,p}^2 \\
&\quad + \sum_{j \in \mathbf{N}_0 \setminus J} e^{-c2^{\alpha j}(t-s)} 2^{2\beta j} |g_j(s,\cdot)|_{V,p}^2 = A(s,t) + B(s,t),
\end{aligned}$$

where $J = \{j \in \mathbf{N}_0 : 2^{\alpha j}(t-s) \leq 1\}$.

Fix $\kappa \in (0, \frac{2\alpha}{p})$. Using Hölder's inequality, we get

$$\begin{aligned} A(s, t) &\leq \sum_{j \in J} 2^{2\beta j} 2^{\kappa j} 2^{-\kappa j} |g_j(s, \cdot)|_{V,p}^2 \\ &\leq \left(\sum_{j \in J} 2^{q\kappa j} \right)^{1/q} \left(\sum_{j \in J} 2^{p\beta j} 2^{-p\kappa j/2} |g_j(s, \cdot)|_{V,p}^p \right)^{2/p} \end{aligned}$$

with $q = \frac{p}{p-2}$. Since

$$\sum_{j \in J} 2^{q\kappa j} \leq C(t-s)^{-q\kappa/\alpha},$$

we have

$$\begin{aligned} A(s, t) &\leq C(t-s)^{-\frac{\kappa}{\alpha}} \left(\sum_{j \in J} 2^{p\beta j} 2^{-p\kappa j/2} |g_j(s, \cdot)|_{V,p}^p \right)^{2/p} \\ &= C(t-s)^{-\frac{\kappa}{\alpha}} \left(\sum_{j=0}^{\infty} 1_{\{(t-s) \leq 2^{-\alpha j}\}} 2^{p\beta j} 2^{-p\kappa j/2} |g_j(s, \cdot)|_{V,p}^p \right)^{2/p}. \end{aligned}$$

So,

$$\begin{aligned} \int_a^b \int_s^b A(s, t)^{p/2} dt ds &\leq C \int_a^b \sum_{j=0}^{\infty} 2^{p\beta j} 2^{-p\kappa j/2} |g_j(s, \cdot)|_{V,p}^p \int_s^{s+2^{-\alpha j}} (t-s)^{-\frac{p\kappa}{2\alpha}} dt ds \\ &\leq C \int_a^b \sum_{j=0}^{\infty} 2^{-\alpha j} 2^{p\beta j} |g_j(s, \cdot)|_{V,p}^p ds = C |g|_{B_{pp}^{\beta-\frac{\alpha}{p}}(E_{a,b}, V)}^p. \end{aligned}$$

Let us consider the sum $B(s, t)$. By Hölder's inequality,

$$B(s, t) \leq \left\{ \sum_{j \in \mathbf{N}_0 \setminus J} e^{-c2^{\alpha j}(t-s)} \right\}^{\frac{1}{q}} \left\{ \sum_{j \in \mathbf{N}_0 \setminus J} e^{-c2^{\alpha j}(t-s)} 2^{\beta p j} |g_j(s, \cdot)|_{V,p}^p \right\}^{\frac{2}{p}}$$

with $q = \frac{p}{p-2}$. Since $e^{-c2^{\alpha j}(t-s)}$ is decreasing in j ,

$$\sum_{j \in \mathbf{N}_0 \setminus J} e^{-c2^{\alpha j}(t-s)} \leq \int_{\{2^{\alpha r}(t-s) \geq 1\}} e^{-c2^{-\alpha} 2^{\alpha r}(t-s)} dr \leq C.$$

Therefore,

$$\begin{aligned} \int_a^b \int_s^b B(s, t)^{p/2} dt ds &\leq C \int_a^b \sum_{j=0}^{\infty} \int_s^b e^{-c2^{\alpha j}(t-s)} dt 2^{\beta p j} |g_j(s, \cdot)|_{V,p}^p ds \\ &\leq C \int_a^b \sum_{j=0}^{\infty} 2^{-\alpha j} 2^{\beta p j} |g_j(s, \cdot)|_{V,p}^p ds. \end{aligned}$$

Finally,

$$\begin{aligned} |\mathcal{I}g|_{\dot{H}_p^\beta(\tilde{E}_{a,b},V)}^p &\leq C \left[\int_a^b \int_s^b A(s,t)^{p/2} dt ds + \int_a^b \int_s^b B(s,t)^{p/2} dt ds \right] \\ &\leq C \int_a^b \sum_{j=0}^{\infty} 2^{-\alpha_j} 2^{\beta p j} |g_j(s, \cdot)|_{V,p}^p ds \leq C |g|_{B_{pp}^{\beta-\frac{\alpha}{p}}(E_{a,b},V)}^p. \end{aligned}$$

The proposition is proved.

5. PROOF OF PROPOSITION 2

In the proof we follow an idea communicated by N.V. Krylov.

5.1. Auxiliary results. We start with

Lemma 2. *Let $\delta \in (0, 1)$, $l \in (-d, \delta)$. Assume that a function $F : \mathbf{R}_0^d \rightarrow \mathbf{R}$ satisfies the inequalities*

$$|F(\xi)| \leq C|\xi|^l, |\nabla F(\xi)| \leq C|\xi|^{l-1}, \xi \in \mathbf{R}_0^d.$$

Then

$$|\partial^\delta F(\xi)| \leq C|\xi|^{l-\delta}, \xi \in \mathbf{R}_0^d.$$

Proof. For any $\xi \in \mathbf{R}_0^d$,

$$\begin{aligned} |\partial^\delta F(\xi)| &= C \left| \int [F(\xi + y) - F(\xi)] \frac{dy}{|y|^{d+\delta}} \right| \\ &\leq C \int_{|y| > \frac{1}{2}|\xi|} [|F(\xi + y)| + |F(\xi)|] \frac{dy}{|y|^{d+\delta}} \\ &\quad + C \int_{|y| \leq \frac{1}{2}|\xi|} \int_0^1 |\nabla F(\xi + sy)| \frac{ds dy}{|y|^{d+\delta-1}}, \end{aligned}$$

where the constant $C = C(\delta)$.

Changing the variable of integration, $y = |\xi|\bar{y}$, we have

$$\begin{aligned} \int_{|y| > \frac{1}{2}|\xi|} |F(\xi + y)| \frac{dy}{|y|^{d+\delta}} &\leq C \int |\xi + y|^l \frac{dy}{|y|^{d+\delta}} \\ &= C|\xi|^{l-\delta} \int_{|\bar{y}| \geq \frac{1}{2}} \left| \frac{\xi}{|\xi|} + \bar{y} \right|^l \frac{d\bar{y}}{|\bar{y}|^{d+\delta}} \\ &\leq C|\xi|^{l-\delta} \sup_{|w|=1} \int_{|\bar{y}| \geq \frac{1}{2}} |w + \bar{y}|^l \frac{d\bar{y}}{|\bar{y}|^{d+\delta}}. \end{aligned}$$

Obviously,

$$\int_{|y| \geq \frac{1}{2}|\xi|} |F(\xi)| \frac{dy}{|y|^{d+\delta}} \leq C|\xi|^l \int_{|y| \geq \frac{1}{2}|\xi|} \frac{dy}{|y|^{d+\delta}} \leq C|\xi|^{l-\delta}.$$

If $|y| \leq \frac{1}{2}|\xi|$, $s \in (0, 1)$, then $|\xi + sy| \geq |\xi| - s|y| \geq \frac{1}{2}|\xi|$ and

$$\begin{aligned} \int_{|y| \leq \frac{1}{2}|\xi|} \int_0^1 |\nabla F(\xi + sy)| \frac{ds dy}{|y|^{d+\delta-1}} &\leq C \int_{|y| \leq \frac{1}{2}|\xi|} \int_0^1 |\xi + sy|^{l-1} \frac{ds dy}{|y|^{d+\delta-1}} \\ &\leq C \int_{|y| \leq \frac{1}{2}|\xi|} |\xi|^{l-1} \frac{dy}{|y|^{d+\delta-1}} \leq C |\xi|^{l-\delta}. \end{aligned}$$

□

We will need some facts about maximal and sharp functions as well (see [13]).

For each $(s, z) \in \mathbf{R}^{d+1}$ and $\delta > 0$ we consider a family of open sets $B(s, z; \delta)$ of the form

$$B(s, z; \delta) = (s - \delta^\alpha, s + \delta^\alpha) \times (z_1 - \delta, z_1 + \delta) \times \dots \times (z_d - \delta, z_d + \delta).$$

Let \mathbb{Q}_δ be the family of all $B(s, z; \delta)$, $(s, z) \in \mathbf{R}^{d+1}$, and $\mathbb{Q} = \cup_{\delta>0} \mathbb{Q}_\delta$. The collection \mathbb{Q} satisfies the basic assumptions in [13] (see I.2.3 in [13]).

Let $h \in L_1(\mathbf{R}^{d+1})$. For the rectangle $B \in \mathbb{Q}$ we set

$$\begin{aligned} h_B &= \frac{1}{\text{mes } B} \int_B h(s, y) ds dy, \\ h_B^\# &= \frac{1}{\text{mes } B} \int_B |h(s, y) - h_B| ds dy. \end{aligned}$$

Let

$$\begin{aligned} Mh(t, x) &= \sup_{\delta>0} \frac{1}{\text{mes } B(t, x; \delta)} \int_{B(t, x; \delta)} |h(s, y)| ds dy, \\ h^\#(t, x) &= \sup_{B \in \mathbb{Q}, (t, x) \in B} h_B^\#, (t, x) \in \mathbf{R}^{d+1}. \end{aligned}$$

In the definition of $h^\#$ the supremum is taken over all $B \in \mathbb{Q} = \cup_{\delta>0} \mathbb{Q}_\delta$ such that $(t, x) \in B$. The functions Mh and $h^\#$ are called the maximal and sharp functions of h .

By Hölder's inequality for $h \in L_2(\mathbf{R}^{d+1})$,

$$(5.1) \quad \left(h_B^\#\right)^2 \leq \frac{1}{\text{mes } B} \int_B h^2(s, y) ds dy,$$

$$(5.2) \quad \left(h_B^\#\right)^2 \leq \frac{1}{(\text{mes } B)^2} \int_B \int_B (h(s, y) - h(u, z))^2 du dz ds dy.$$

We will also use the maximal functions defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\text{mes } B_r(0)} \int_{B_r(x)} |f(y)| dy,$$

where $B_r(x) = \{y \in \mathbf{R}^d : |y - x| < r\}$.

As it is well known ([13], Theorem IV.2.2,), for $h \in L_p(\mathbf{R}^{d+1})$, $p > 1$, the following norms are equivalent:

$$(5.3) \quad |h|_p \sim |Mh|_p \sim |h^\#|_p.$$

Also, for $h \in L_p(\mathbf{R}^d)$, $p > 1$,

$$(5.4) \quad |h|_p \sim |\mathcal{M}h|_p.$$

Lemma 3. *Let $f \in C_0^\infty(\mathbf{R}^d)$ and v be a continuously differentiable function on \mathbf{R}^d such that $\lim_{|z| \rightarrow \infty} |v(z)| = 0$. Let $R, R_1 \geq 0$, $x, y \in \mathbf{R}^d$, $|x - y| \leq R_1$ and $f(z) = 0$ if $|y - z| \leq R$.*

Then

$$|(f * v)(y)| \leq C [\mathcal{M}f^2(x)]^{\frac{1}{2}} \int_R^\infty (R_1 + \rho)^d \Phi(\rho) d\rho,$$

where the constant $C = C(d)$ and

$$\Phi(\rho) = \left(\int_{|w|=1} (\nabla v(\rho w), w)^2 dw \right)^{\frac{1}{2}},$$

where dw is the counting measure on $\{-1, 1\}$ if $d = 1$, and dw is the Lebesgue measure if $d \geq 2$.

Proof. Integrating by parts, we have

$$\begin{aligned} \int f(y - z)v(z)dz &= \int_R^\infty \int_{|w|=1} f(y - \rho w)v(\rho w)\rho^{d-1}dw d\rho \\ &= \int_R^\infty \int_{|w|=1} v(\rho w) \frac{d}{d\rho} \int_R^\rho f(y - rw)r^{d-1}dr dw d\rho \\ &= \int_{|w|=1} \left[v(\rho w) \int_R^\rho f(y - rw)r^{d-1}dr \right] \Big|_R^\infty dw \\ &\quad - \int_R^\infty \int_{|w|=1} \int_R^\rho f(y - rw)r^{d-1}dr (\nabla v(\rho w), w) dw d\rho \\ &= - \int_R^\infty \int_{|w|=1} \int_R^\rho f(y - rw)r^{d-1}dr (\nabla v(\rho w), w) dw d\rho. \end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
|(f * v)(y)| &\leq \int_R^\infty \left(\int_R^\rho \int_{|w|=1} f^2(y - rw) r^{d-1} dw dr \right)^{\frac{1}{2}} \left(\int_R^\rho r^{d-1} dr \right)^{\frac{1}{2}} \Phi(\rho) d\rho \\
&\leq C \int_R^\infty \left(\int_{B_\rho(y)} f^2(z) dz \right)^{\frac{1}{2}} \rho^{\frac{d}{2}} \Phi(\rho) d\rho \\
&\leq C \int_R^\infty \left(\int_{B_{R_1+\rho}(x)} f^2(z) dz \right)^{\frac{1}{2}} \rho^{\frac{d}{2}} \Phi(\rho) d\rho \\
&\leq C \int_R^\infty (R_1 + \rho)^{\frac{d}{2}} \rho^{\frac{d}{2}} \left(\sup_{\rho>0} (R_1 + \rho)^{-d} \int_{B_{R_1+\rho}(x)} f^2(z) dz \right)^{\frac{1}{2}} \Phi(\rho) d\rho \\
&\leq C [\mathcal{M}f^2(x)]^{\frac{1}{2}} \int_R^\infty (R_1 + \rho)^d \Phi(\rho) d\rho.
\end{aligned}$$

□

5.2. Proof of Proposition 2. ¹⁰. Since $(I - \Delta)^{\beta/2} : H_p^s \rightarrow H_p^{s-\beta/2}$, $s \in \mathbf{R}$, is an isomorphism (see [12]), it is enough to prove the first inequality for $\beta = 0$. Also, it is enough to consider $g \in C_0^\infty(\mathbf{R}^{d+1}, V)$, the space of smooth V -valued functions on \mathbf{R}^{d+1} with compact support.

Let us introduce the function

$$\tilde{\psi}^{(\alpha)}(t, \xi) = \psi^{(\alpha)}\left(t, \frac{\xi}{|\xi|}\right), \quad \xi \in \mathbf{R}_0^d = \{\xi \in \mathbf{R}^d : \xi \neq 0\}.$$

Obviously, if $\alpha \neq 1$,

$$(5.5) \quad \psi^{(\alpha)}(t, \xi) = |\xi|^\alpha \tilde{\psi}^{(\alpha)}(t, \xi).$$

Since

$$\begin{aligned}
(w, \xi) \ln |(w, \xi)| &= |\xi| (w, \frac{\xi}{|\xi|}) \ln |(w, \frac{\xi}{|\xi|})| |\xi| \\
&= |\xi| (w, \frac{\xi}{|\xi|}) \ln |(w, \frac{\xi}{|\xi|})| + |\xi| (w, \frac{\xi}{|\xi|}) \ln |\xi|
\end{aligned}$$

and $\int_{|w|=1} w m^{(1)}(t, w) dw = 0$, the equality (5.5) holds for $\alpha = 1$ as well. By Assumption **B**,

$$\operatorname{Re} \tilde{\psi}^{(\alpha)}(t, \xi) \leq -\mu < 0, \quad t \in \mathbf{R}, \quad \xi \in \mathbf{R}_0^d.$$

Let $p = 2$ and $g \in H_2^0(E_{a,b}, V)$. Then, by Parseval's equality,

$$\begin{aligned}
|\partial^{\alpha/2} \mathcal{I}g|_{H_2^0(\tilde{E}_{a,b}, V)}^2 &= \int_a^b \int_a^t \int |\partial^{\alpha/2} \mathcal{I}g(s, t, x)|_V^2 dx ds dt \\
&= \int_a^b \int_a^t \int ||\xi|^{\alpha/2} e^{|\xi|^\alpha \int_s^t \tilde{\psi}^{(\alpha)}(r, \xi) dr} \mathcal{F}g(s, \xi)|_V^2 d\xi ds dt \\
&\leq \int_a^b \int_a^t \int |\xi|^\alpha e^{-2\mu|\xi|^\alpha(t-s)} |\mathcal{F}g(s, \xi)|_V^2 d\xi ds dt \\
&= \int \int_a^b \int_s^b |\xi|^\alpha e^{-2\mu|\xi|^\alpha(t-s)} |\mathcal{F}g(s, \xi)|_V^2 dt ds d\xi \\
&\leq (2\mu)^{-1} \int_a^b \int |\mathcal{F}g(s, \xi)|_V^2 d\xi ds \\
(5.6) \quad &= (2\mu)^{-1} |g|_{H_2^0(E_{a,b}, V)}^2.
\end{aligned}$$

2°. Let $p > 2$. We extend the functions $g \in H_p^0(E_{a,b}, V)$ by zero outside the interval $[a, b]$ if necessary. Obviously, the extended functions belong to $H_p^0(E, V)$, where $E = E_{-\infty, \infty} = \mathbf{R}^{d+1}$.

For $g \in H_p^0(E, V)$ we denote

$$\begin{aligned}
Gg(s, y) &= \left\{ \int_{-\infty}^s \left| \int \partial^{\alpha/2} G_{u,s}(y - y') g(u, y') dy' \right|_V^2 du \right\}^{1/2} \\
&= \left\{ \int_{-\infty}^s \left| \int G_{u,s}(y - y') \partial^{\alpha/2} g(u, y') dy' \right|_V^2 du \right\}^{1/2}.
\end{aligned}$$

Note that by triangle inequality in $L_2((-\infty, s], V)$ we have for $g_1, g_2 \in H_p^0(E, V)$,

$$\begin{aligned}
(5.7) \quad G(g_1 + g_2)(s, y) &\leq Gg_1(s, y) + Gg_2(s, y), \\
|G(g_1 + g_2)(s, y) - Gg_1(s, y)| &\leq Gg_2(s, y).
\end{aligned}$$

According to (5.3) and (5.4) it is enough to prove that there is a constant C such that for all $g \in H_p^0(E, V)$, $(t, x) \in \mathbf{R}^{d+1}$

$$(5.8) \quad (Gg)^\#(t, x) \leq C(\mathcal{M}_t \mathcal{M}_x |g|_V^2(t, x))^{1/2},$$

where \mathcal{M}_t and \mathcal{M}_x denote the maximal functions defined using the balls in \mathbf{R} and \mathbf{R}^d and

$$(Gg)^\#(t, x) = \sup_{B \in \mathbb{Q}, (t, x) \in B} \frac{1}{\text{mes}(B)} \int_B |Gg(s, y) - (Gg)_B| ds dy.$$

Since $B \in \mathbb{Q}$ is of the form

$$\begin{aligned} B &= (s_0 - \delta^\alpha, s_0 + \delta^\alpha) \times (z_1 - \delta, z_1 + \delta) \times \dots \times (z_d - \delta, z_d + \delta) \\ &= (\tilde{s}_0, z) + \tilde{B}(0, 0; \delta), \end{aligned}$$

with $\tilde{s}_0 = s_0 + \delta^\alpha$, $\tilde{B}(0, 0; \delta) = (-2\delta^\alpha, 0) \times (-\delta, \delta)^d$, it is straightforward to verify that

$$\begin{aligned} & \frac{1}{\text{mes}(B)} \int_B |Gg(s, y) - (Gg)_B| ds dy \\ &= \frac{1}{\text{mes}(Q_0)} \int_{Q_0} |Gg(\tilde{s}_0 + \delta^\alpha s, z + \delta y) - (Gg(\tilde{s}_0 + \delta^\alpha \cdot, z + \delta \cdot))_{Q_0}| ds dy, \end{aligned}$$

where $Q_0 = \tilde{B}(0, 0; 1)$.

Changing the variable of integration, $u = \tilde{s}_0 + \delta^\alpha s$, $y' = z + \delta y$, we see that

$$\begin{aligned} & Gg(\tilde{s}_0 + \delta^\alpha t, z + \delta x) \\ &= \left\{ \int_{-\infty}^{\tilde{s}_0 + \delta^\alpha t} \left| \int \partial^{\alpha/2} G_{u, \tilde{s}_0 + \delta^\alpha t}(z + \delta x - y') g(u, y') dy' \right|_V^2 du \right\}^{1/2} \\ &= \delta^{\frac{\alpha}{2} + d} \left\{ \int_{-\infty}^t \left| \int \partial^{\alpha/2} G_{\tilde{s}_0 + \delta^\alpha s, \tilde{s}_0 + \delta^\alpha t}(\delta(x - y)) g(\tilde{s}_0 + \delta^\alpha s, z + \delta y) dy \right|_V^2 ds \right\}^{1/2} \\ &= \left\{ \int_{-\infty}^t \left| \int \partial^{\alpha/2} G_{s, t}^{\tilde{s}_0, \delta}(x - y) g(\tilde{s}_0 + \delta^\alpha s, z + \delta y) dy \right|_V^2 ds \right\}^{1/2}, \end{aligned}$$

where

$$G_{s, t}^{s_0, \delta}(x) = \mathcal{F}^{-1} \left(\exp \left\{ \int_s^t \psi^{(\alpha)}(s_0 + \delta^\alpha r, \xi) dr \right\} \right) (x)$$

with

$$\begin{aligned} \psi^{(\alpha)}(s_0 + \delta^\alpha t, \xi) &= i(b(s_0 + \delta^\alpha t), \xi) 1_{\alpha=1} - \sum_{i, j=1}^d B^{ij}(s_0 + \delta^\alpha t) \xi_i \xi_j 1_{\alpha=2} \\ &\quad - C \int_{S^{d-1}} |(w, \xi)|^\alpha \left[1 - i \left(\tan \frac{\alpha\pi}{2} \text{sgn}(w, \xi) 1_{\alpha \neq 1} \right. \right. \\ &\quad \left. \left. - \frac{2}{\pi} \text{sgn}(w, \xi) \ln |(w, \xi)| 1_{\alpha=1} \right) \right] m^{(\alpha)}(s_0 + \delta^\alpha t, w) dw. \end{aligned}$$

Note that for every $\tilde{s}_0 \in \mathbf{R}^d$, $\delta > 0$, the coefficients $b(s_0 + \delta^\alpha t)$, $B^{ij}(s_0 + \delta^\alpha t)$, $m^{(\alpha)}(s_0 + \delta^\alpha t, w)$, $t \in \mathbf{R}$, $w \in S^{d-1}$, satisfy the assumptions **A, B**

with the same constants K and μ . Therefore for (5.8) it is enough to show the inequality

$$(5.9) \quad (Gg)_{Q_0}^\# \leq C (\mathcal{M}_t \mathcal{M}_x |g(t, x)|_V^2)^{1/2}, (t, x) \in Q_0,$$

with

$$Q_0 = \tilde{B}(0, 0; 1) = \{(t, x) \in [-2, 0] \times [-1, 1]^d\}$$

We consider the following three cases:

- (1) $g(t, x) = 0, (t, x) \notin [-12, 12] \times B_{3\sqrt{d}}(0);$
- (2) $g(t, x) = 0, (t, x) \notin [-12, 12] \times \mathbf{R}^d;$
- (3) $g(t, x) = 0, t \geq -8, x \in \mathbf{R}^d.$

For the estimates of the derivatives of $G_{u,s}(x)$ the following representation is helpful. For $u < s, x \in \mathbf{R}^d, j, k = 1, \dots, d,$

$$(5.10) \quad \begin{aligned} \partial_j \partial_k \partial^{\alpha/2} G_{u,s}(x) &= (s-u)^{-\frac{d}{\alpha}-\frac{1}{2}-\frac{2}{\alpha}} F_{u,s}^{j,k} \left((s-u)^{-\frac{1}{\alpha}} x \right), \\ \partial_j \partial^{\alpha/2} G_{u,s}(x) &= (s-u)^{-\frac{d}{\alpha}-\frac{1}{2}-\frac{1}{\alpha}} F_{u,s}^j \left((s-u)^{-\frac{1}{\alpha}} x \right), \\ \partial_s \partial^{\frac{\alpha}{2}} G_{u,s}(x) &= (s-u)^{-\frac{d}{\alpha}-\frac{3}{2}} \bar{F}_{u,s} \left((s-u)^{-\frac{1}{\alpha}} x \right), \\ \partial_j \partial_s \partial^{\frac{\alpha}{2}} G_{u,s}(x) &= (s-u)^{-\frac{d}{\alpha}-\frac{3}{2}-\frac{1}{\alpha}} \bar{F}_{u,s}^j \left((s-u)^{-\frac{1}{\alpha}} x \right), \end{aligned}$$

with

$$\begin{aligned} F_{u,s}^{j,k} &= \mathcal{F}^{-1} \left\{ -\xi_j \xi_k |\xi|^{\frac{\alpha}{2}} \exp \left\{ -|\xi|^\alpha \frac{1}{(s-u)} \int_u^s \tilde{\psi}^{(\alpha)}(r, \xi) dr \right\} \right\}, \\ F_{u,s}^j &= \mathcal{F}^{-1} \left\{ i \xi_k |\xi|^{\frac{\alpha}{2}} \exp \left\{ -|\xi|^\alpha \frac{1}{(s-u)} \int_u^s \tilde{\psi}^{(\alpha)}(r, \xi) dr \right\} \right\}, \\ \bar{F}_{u,s} &= \mathcal{F}^{-1} \left\{ -|\xi|^{\frac{3}{2}\alpha} \tilde{\psi}^{(\alpha)}(s, \xi) \exp \left\{ -|\xi|^\alpha \frac{1}{(s-u)} \int_u^s \tilde{\psi}^{(\alpha)}(r, \xi) dr \right\} \right\}, \\ \bar{F}_{u,s}^j &= \mathcal{F}^{-1} \left\{ -i \xi_j |\xi|^{\frac{3}{2}\alpha} \tilde{\psi}^{(\alpha)}(s, \xi) \exp \left\{ -|\xi|^\alpha \frac{1}{(s-u)} \int_u^s \tilde{\psi}^{(\alpha)}(r, \xi) dr \right\} \right\}. \end{aligned}$$

By definition of the inverse Fourier transform, all functions $F_{u,s}^{j,k}, F_{u,s}^j, \bar{F}_{u,s}, \bar{F}_{u,s}^j$ are uniformly bounded.

3⁰. First, we prove that in the case (1)

$$(5.11) \quad \int_{Q_0} (Gg)(s, y)^2 ds dy \leq C \mathcal{M}_t \mathcal{M}_x |g(t, x)|_V^2$$

for all $(t, x) \in Q_0$.

Repeating the proof of (5.6), we have

$$\begin{aligned}
\int_{Q_0} (Gg)^2(s, y) ds dy &\leq \int_{-\infty}^{\infty} \int_{-\infty}^s \int |\xi|^\alpha e^{-2\mu|\xi|^\alpha(s-u)} |\mathcal{F}g(u, \xi)|_V^2 d\xi du ds \\
&\leq (2\mu)^{-1} \int_{-\infty}^{\infty} \int |g(u, y)|_V^2 dy du \\
&= (2\mu)^{-1} \int_{-12}^{12} \int_{B_{2\sqrt{d}}(0)} |g(u, y)|_V^2 dy du.
\end{aligned}$$

Now for every $(t, x) \in Q_0$,

$$\begin{aligned}
&\int_{-12}^{12} \int_{B_{3\sqrt{d}}(0)} |g(u, y)|_V^2 dy du \\
&\leq \text{mes}(B_{5\sqrt{d}}(0)) \int_{-12}^{12} \frac{1}{\text{mes}(B_{5\sqrt{d}}(x))} \int_{B_{5\sqrt{d}}(x)} |g(u, y)|_V^2 dy du \\
&\leq \text{mes } B_{5\sqrt{d}}(0) \int_{-12}^{12} \mathcal{M}_x |g(u, x)|_V^2 du \\
&\leq C \mathcal{M}_t \mathcal{M}_x |g(t, x)|_V^2
\end{aligned}$$

and (5.11) is proven.

⁴⁰. Now we prove that (5.11) holds in the case (2) as well. Since (5.11) holds for $g(t, z) = 0$, $(t, z) \notin [-12, 12] \times B_{3\sqrt{d}}(0)$, it is enough to consider $g(t, z)$ such that $g(t, z) = 0$ if $|t| > 12$ or $|z| \leq 2\sqrt{d}$. By Minkowski's inequality,

$$\begin{aligned}
(Gg)^2(s, y) &= \int_{-\infty}^s \left| \int \partial^{\alpha/2} G_{u,s}(y - y') g(u, y') dy' \right|_V^2 du \\
&\leq \int_{-12}^s \left(\int |\partial^{\alpha/2} G_{u,s}(y - y')| |g(u, y')|_V dy' \right)^2 du.
\end{aligned}$$

According to Lemma 3 (in our case $R = \sqrt{d}$, $R_1 = 2\sqrt{d}$),

$$\begin{aligned}
&\left(\int |\partial^{\alpha/2} G_{u,s}(y - y')| |g(u, y')|_V dy' \right)^2 \leq C \mathcal{M}_x |g(u, x)|_V^2 \times \\
&\quad \times \left(\int_{\sqrt{d}}^{\infty} (2\sqrt{d} + \rho)^d \left(\int_{|w|=1} \sum_{j=1}^d |\partial_j \partial^{\alpha/2} G_{u,s}(\rho w)|^2 dw \right)^{1/2} d\rho \right)^2 \\
&\leq C \mathcal{M}_x |g(u, x)|_V^2 \kappa(u, s),
\end{aligned}$$

where

$$\kappa(u, s) = \left(\int_1^\infty \rho^d \left(\int_{|w|=1} \sum_{j=1}^d |\partial_j \partial^{\alpha/2} G_{u,s}(\rho w)|^2 dw \right)^{1/2} d\rho \right)^2.$$

By (5.10),

$$\kappa(u, s) = (s-u)^{-\frac{2d}{\alpha}-1-\frac{2}{\alpha}} \left(\int_1^\infty \rho^d \left(\int_{|w|=1} \sum_{j=1}^d |F_{u,s}^j(\rho(s-u)^{-\frac{1}{\alpha}} w)|^2 dw \right)^{1/2} d\rho \right)^2.$$

Changing the variable of integration, $\rho(s-u)^{-\frac{1}{\alpha}} = r$, and using Hölder's inequality, we get

$$\begin{aligned} \kappa(u, s) &= (s-u)^{-1} \left(\int_{(s-u)^{-\frac{1}{\alpha}}}^\infty r^d \left(\int_{|w|=1} \sum_{j=1}^d [F_{u,s}^j(rw)]^2 dw \right)^{1/2} dr \right)^2 \\ &\leq (s-u)^{-1} \int_{(s-u)^{-\frac{1}{\alpha}}}^\infty r^{-1-\frac{\alpha}{2}} dr \int_0^\infty r^{2d+1+\frac{\alpha}{2}} \int_{|w|=1} \sum_{j=1}^d [F_{u,s}^j(rw)]^2 dw dr \\ &\leq C(s-u)^{-\frac{1}{2}} \int \sum_{j=1}^d \left[|x|^{\frac{d}{2}+1+\frac{\alpha}{4}} F_{u,s}^j(x) \right]^2 dx. \end{aligned}$$

Hence, by Parseval's equality,

$$\kappa(s, u) \leq C(s-u)^{-\frac{1}{2}} \int \sum_{j=1}^d \left| \partial^{\frac{d}{2}+1+\frac{\alpha}{4}} \mathcal{F} F_{u,s}^j(\xi) \right|^2 d\xi.$$

Due to our assumptions **A**, **B** and Lemma 2, the last integral is finite. Therefore

$$\begin{aligned} \int_{Q_0} (Gg)^2 ds dy &\leq C \int_{-2}^0 \int_{-12}^s (s-u)^{-\frac{1}{2}} \mathcal{M}_x |g(u, x)|_V^2 du ds \\ &= C \left(\int_{-12}^{-2} \int_{-2}^0 l(s, u, x) ds du + \int_{-2}^0 \int_u^0 l(s, u, x) ds du \right) \\ (5.12) \quad &\leq C \int_{-12}^0 \mathcal{M}_x |g(u, x)|_V^2 du \leq C \mathcal{M}_t \mathcal{M}_x |g(t, x)|_V^2 \end{aligned}$$

for all $(t, x) \in Q_0$ with

$$l(s, u, x) = (s-u)^{-\frac{1}{2}} \mathcal{M}_x |g(u, x)|_V^2.$$

5⁰. We will show that in the case (3)

$$(5.13) \quad \int_{Q_0} |Gg(s, y) - Gg(t', x')|^2 ds dy \leq C \mathcal{M}_t \mathcal{M}_x |g|_V^2(t, x)$$

with all $(t, x), (t', x') \in Q_0$. We estimate the Lipschitz constant of Gg in t and x . Obviously, for each $(s, y), (t', x') \in Q_0$

$$(5.14) \quad |Gg(s, y) - Gg(t', x')| \leq C \left(\sup_{(s, y) \in Q_0} |\nabla Gg(s, y)| + \sup_{(s, y) \in Q_0} |\partial_s Gg(s, y)| \right).$$

First we estimate $|\nabla Gg(s, y)|$ in (5.14). Let $\varphi \in C_0^\infty(\mathbf{R}^d)$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $|x| \leq 2\sqrt{d}$, $\varphi(x) = 0$ if $|x| < 3\sqrt{d}$, and

$$\begin{aligned} g_2(u, y') &= g(u, y')\varphi(y'), \\ g_1(u, y') &= g(u, y')(1 - \varphi(y')), \quad (u, y') \in \mathbf{R}^{d+1}. \end{aligned}$$

Since $g(u, y') = 0$ if $u \geq -8$, applying Hölder's and Minkowski's inequalities, we derive for $s \in [-2, 0], |y| \leq 1$,

$$\begin{aligned} |\nabla Gg(s, y)|^2 &\leq \int_{-\infty}^{-8} \left| \int \nabla \partial^{\alpha/2} G_{u,s}(y - y') g(u, y') dy' \right|_V^2 du \\ &\leq 2 \int_{-\infty}^{-8} \left(\int_{|y'| > 2\sqrt{d}} |\nabla \partial^{\alpha/2} G_{u,s}(y - y')| |g_1(u, y')|_V dy' \right)^2 du \\ &\quad + 2 \int_{-\infty}^{-8} \left(\int_{|y'| \leq 3\sqrt{d}} |\nabla \partial^{\alpha/2} G_{u,s}(y - y')| |g_2(u, y')|_V dy' \right)^2 du \\ &= 2(A_1(s, y) + A_2(s, y)). \end{aligned}$$

For any $(t, x) \in Q_0$, according to (5.10) and Lemma 1 (applied for $d = 1$),

$$\begin{aligned} A_2(s, y) &\leq \int_{-\infty}^{-8} \sup_{|z| \leq 4\sqrt{d}} |\nabla \partial^{\alpha/2} G_{u,s}(z)|^2 \left(\int_{|y'| \leq 3\sqrt{d}} |g(u, y')|_V dy' \right)^2 du \\ &\leq C \int_{-\infty}^{-8} \sup_{|z| \leq 4\sqrt{d}} |\nabla \partial^{\alpha/2} G_{u,s}(z)|^2 \left(\frac{1}{\text{mes } B_{4\sqrt{d}}(x)} \int_{|x-y'| \leq 4\sqrt{d}} |g(u, y')|_V^2 dy' \right) du \\ &\leq C \int_{-\infty}^{-8} (s - u)^{-\frac{2d}{\alpha} - 1 - \frac{2}{\alpha}} \mathcal{M}_x(|g|_V^2)(u, x) du \leq C \mathcal{M}_t \mathcal{M}_x(|g|_V^2)(t, x) \end{aligned}$$

According to Lemma 3 (in our case $R = \sqrt{d}$ and $R_1 = 2\sqrt{d}$),

$$\begin{aligned} & \left(\int_{|y'| \geq 2\sqrt{d}} |\nabla \partial^{\alpha/2} G_{u,s}(y - y')| |g_1(u, y')|_V dy' \right)^2 \leq C \mathcal{M}_x |g(u, x)|_V^2 \times \\ & \quad \times \left(\int_{\sqrt{d}}^{\infty} (2\sqrt{d} + \rho)^d \left(\int_{|w|=1} \sum_{j=1}^d |\partial_j \nabla \partial^{\alpha/2} G_{u,s}(\rho w)|^2 dw \right)^{1/2} d\rho \right)^2 \\ & \leq C \mathcal{M}_x |g(u, x)|_V^2 \tilde{\kappa}(u, s), \end{aligned}$$

where

$$\tilde{\kappa}(u, s) = \left(\int_1^{\infty} \rho^d \left(\int_{|w|=1} \sum_{j=1}^d |\partial_j \nabla \partial^{\alpha/2} G_{u,s}(\rho w)|^2 dw \right)^{1/2} d\rho \right)^2.$$

By (5.10) and Hölder's inequality

$$\begin{aligned} \tilde{\kappa}(u, s) &= (s - u)^{-p} \left(\int_1^{\infty} \rho^d \left(\int_{|w|=1} \sum_{j,k=1}^d \left[F_{u,s}^{j,k}(\rho(s - u)^{-\frac{1}{\alpha}} w) \right]^2 dw \right)^{1/2} d\rho \right)^2 \\ &\leq (s - u)^{-p} \int_1^{\infty} \rho^{-2} d\rho \int_1^{\infty} \rho^{2d+2} \int_{|w|=1} \sum_{j,k=1}^d \left[F_{u,s}^{j,k}(\rho(s - u)^{-\frac{1}{\alpha}} w) \right]^2 dw d\rho \\ &= (s - u)^{-p} \int_{|x| \geq 1} |x|^{d+3} \sum_{j,k=1}^d \left[F_{u,s}^{j,k}((s - u)^{-\frac{1}{\alpha}} x) \right]^2 dx \end{aligned}$$

with $p = \frac{2d+4}{\alpha} + 1$.

Changing the variable of integration, $y = (s - u)^{-\frac{1}{\alpha}} x$, we get by Parseval's equality

$$\begin{aligned} \tilde{\kappa}(u, s) &\leq (s - u)^{-1-\frac{1}{\alpha}} \int |y|^{d+3} \sum_{j,k=1}^d \left[F_{u,s}^{j,k}(y) \right]^2 dy \\ &= (s - u)^{-1-\frac{1}{\alpha}} \int \sum_{j,k=1}^d \left| \partial^{\frac{d+3}{2}} \mathcal{F} F_{u,s}^{j,k}(\xi) \right|^2 d\xi. \end{aligned}$$

Due to our assumptions and Lemma 2, the last integral is finite. Hence,

$$\tilde{\kappa}(u, s) \leq C(s - u)^{-1-\frac{1}{\alpha}}$$

and for $(s, y) \in Q_0$,

$$A_1(s, y) \leq \int_{-\infty}^{-8} \mathcal{M}_x |g|_V^2(u, x) \tilde{\kappa}(u, s) du \leq C \int_{-\infty}^{-8} \mathcal{M}_x |g|_V^2(u, x) (s - u)^{-1-\frac{1}{\alpha}} du.$$

Therefore by Lemma 1 (in the case $d = 1$), for $(s, y) \in Q_0, (t, x) \in Q_0$,

$$(5.15) \quad |\nabla Gg(s, y)|^2 \leq A_1(s, y) + A_2(s, y) \leq C\mathcal{M}_t\mathcal{M}_x|g|_V^2(t, x).$$

Now we estimate $|\partial_s Gg(s, y)|$. Applying Hölder's and Minkowski's inequalities, we get for $(s, y) \in Q_0$,

$$\begin{aligned} [\partial_s G(s, y)]^2 &\leq \int_{-\infty}^{-8} \left| \int \partial_s \partial^{\alpha/2} G_{u,s}(y - y') g(u, y') dy' \right|_V^2 du \\ &\leq 2 \int_{-\infty}^{-8} \left(\int_{|y'| > 2\sqrt{d}} |\partial_s \partial^{\alpha/2} G_{u,s}(y - y')| |g_1(u, y')|_V dy' \right)^2 du \\ &\quad + \int_{-\infty}^{-8} \left(\int_{|y'| \leq 3\sqrt{d}} |\partial_s \partial^{\alpha/2} G_{u,s}(y - y')| |g_2(u, y')|_V dy' \right)^2 du \\ &= 2B_1(s, y) + 2B_2(s, y). \end{aligned}$$

According to Lemma 3,

$$\begin{aligned} &\left(\int_{|y'| > 2\sqrt{d}} |\partial_s \partial^{\alpha/2} G_{u,s}(y - y')| |g_1(u, y')|_V dy' \right)^2 \leq C\mathcal{M}_x|g|_V^2(u, x) \times \\ &\quad \times \left(\int_{\sqrt{d}}^{\infty} (2\sqrt{d} + \rho)^d \left(\int_{|w|=1} \sum_{j=1}^d [\partial_j \partial_s \partial^{\alpha/2} G_{u,s}(\rho w)]^2 dw \right)^{\frac{1}{2}} d\rho \right)^2 \\ &\leq C\bar{\kappa}(u, s)\mathcal{M}_x|g|_V^2(u, x), \end{aligned}$$

where

$$\bar{\kappa}(u, s) = \left(\int_1^{\infty} \rho^d \left(\int_{|w|=1} \sum_{j=1}^d [\partial_j \partial_s \partial^{\alpha/2} G_{u,s}(\rho w)]^2 dw \right)^{\frac{1}{2}} d\rho \right)^2.$$

According to (5.10), we have by Hölder's inequality

$$\begin{aligned} \bar{\kappa}(u, s) &= (s - u)^{-p} \left(\int_1^{\infty} \rho^d \left(\int_{|w|=1} \sum_{j=1}^d [\bar{F}_{u,s}^j(\rho(s - u)^{-\frac{1}{\alpha}} w)]^2 dw \right)^{\frac{1}{2}} d\rho \right)^2 \\ &\leq (s - u)^{-p} \int_1^{\infty} \rho^{-1-\alpha} d\rho \int_1^{\infty} \rho^{2d+1+\alpha} \int_{|w|=1} \sum_{j=1}^d [\bar{F}_{u,s}^j(\rho(s - u)^{-\frac{1}{\alpha}} w)]^2 dw d\rho \\ &\leq C(s - u)^{-p} \int_{|x| \geq 1} |x|^{d+2+\alpha} \sum_{j=1}^d [\bar{F}_{u,s}^j((s - u)^{-\frac{1}{\alpha}} x)]^2 dx, \end{aligned}$$

where $p = \frac{2d}{\alpha} + 3 + \frac{2}{\alpha}$. Changing the variable of integration, $y = (s - u)^{-\frac{1}{\alpha}}x$, we get by Parseval's equality

$$\begin{aligned}\bar{\kappa}(u, s) &\leq C(s - u)^{-2} \int \sum_{j=1}^d [|y|^{\frac{d}{2}+1+\frac{\alpha}{2}} \bar{F}_{u,s}^j(y)]^2 dy \\ &\leq C(s - u)^{-2} \int \sum_{j=1}^d |\partial^{\frac{d}{2}+1+\frac{\alpha}{2}} \mathcal{F} \bar{F}_{u,s}^j(\xi)|^2 d\xi.\end{aligned}$$

Due to our assumptions and Lemma 2, the last integral is finite. Hence,

$$\bar{\kappa}(u, s) \leq C(s - u)^{-2}$$

and, by Lemma 1 ($d = 1$) it follows for $(s, y), (t, x) \in Q_0$,
(5.16)

$$B_1(s, y) \leq C \int_{-\infty}^{-8} (s - u)^{-2} \mathcal{M}_x |g|_V^2(u, x) du \leq C \mathcal{M}_t \mathcal{M}_x |g|_V^2(t, x).$$

For any $(s, y), (t, x) \in Q_0$, according to (5.10) and Lemma 1 ($d = 1$),

$$\begin{aligned}B_2(s, y) &\leq \int_{-\infty}^{-8} \sup_{|z| \leq 4\sqrt{d}} |\partial_s \partial^{\alpha/2} G_{u,s}(z)|^2 \left(\int_{|y'| \leq 3\sqrt{d}} |g(u, y')|_V dy' \right)^2 du \\ &\leq C \int_{-\infty}^{-8} \sup_{|z| \leq 4\sqrt{d}} |\partial_s \partial^{\alpha/2} G_{u,s}(z)|^2 \left(\frac{1}{\text{mes } B_{4\sqrt{d}}(x)} \int_{|x-y'| \leq 4\sqrt{d}} |g(u, y')|_V^2 dy' \right) du \\ &\leq C \int_{-\infty}^{-8} (s - u)^{-\frac{2d}{\alpha}-3} \mathcal{M}_x |g|_V^2(u, x) du \leq C \mathcal{M}_t \mathcal{M}_x (|g|_V^2)(t, x).\end{aligned}$$

Summarizing, we have for all $(s, y), (t, x) \in Q_0$,

$$|\nabla G(s, y)|^2 + [\partial_s G(s, y)]^2 \leq C \mathcal{M}_t \mathcal{M}_x (|g|_V^2)(t, x)|_V^2.$$

Therefore (5.13) follows and we showed that (5.11) holds in the first and second case.

6°. Now we show that (5.11) in the case (2)-(1) and (5.13) in the case (3) imply (5.9). Let φ be a continuous function on \mathbf{R} with all bounded derivatives such that $0 \leq \varphi \leq 1$, $\varphi(s) = 0$ if $-8 \leq s$, $\varphi(s) = 1$ if $s \leq -9$. Let

$$\begin{aligned}g_1(s, y) &= g(s, y)\varphi(s), (s, y) \in \mathbf{R}^{d+1}, \\ g_2 &= g - g_1.\end{aligned}$$

Then by (5.7),

$$\begin{aligned}|Gg - (Gg)_{Q_0}| &\leq |G(g_1 + g_2) - Gg_1| + |Gg_1 - (Gg_1)_{Q_0}| \\ + |(Gg_1)_{Q_0} - (Gg)_{Q_0}| &\leq Gg_2 + (Gg_2)_{Q_0} + |Gg_1 - (Gg_1)_{Q_0}|\end{aligned}$$

and

$$(Gg)_{Q_0}^\# \leq (Gg_1)_{Q_0}^\# + 2(Gg_2)_{Q_0}$$

Now, by (5.1) and (5.2), the required inequality (5.9) follows from (5.11), (5.12) and (5.13). The first assertion of the proposition is proved.

7⁰. The estimates in Besov spaces follow immediately because

$$(\partial^{\alpha/2} \mathcal{I}g)_j = \partial^{\alpha/2} \mathcal{I}g_j$$

and we have shown that

$$\begin{aligned} & \int_a^b \int_{\mathbf{R}^d} \left(\int_a^t |\varphi_j * f(s, t, x)|_V^2 ds \right)^{p/2} dx dt \\ &= \int_a^b \int_{\mathbf{R}^d} \left(\int_a^t |(\partial^{\alpha/2} \mathcal{I}g(s, t, x))_j|_V^2 ds \right)^{p/2} dx dt \\ &= \int_a^b \int_{\mathbf{R}^d} \left(\int_a^t |\partial^{\alpha/2} \mathcal{I}g_j(s, t, x)|_V^2 ds \right)^{p/2} dx dt \\ &\leq C \int_a^b |g_j(s, \cdot)|_{V,p}^p ds. \end{aligned}$$

The proposition is proved.

REFERENCES

- [1] Bergh, J. and Löfström, J., *Interpolation Spaces. An Introduction*, Springer Verlag, 1976.
- [2] Grigelionis B., Reduced stochastic equations of the nonlinear filtering of random processes, *Lithuanian Math. J.*, **16** (1976) 348–358.
- [3] Jacod, J., *Calcul Stochastique et Problèmes de Martingales*, *Lecture Notes in Mathematics*, 714, Springer Verlag, Berlin New York, 1979.
- [4] Kim, Ildoo and Kim, Kyeong-Hun, A generalization of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta)^{\alpha/2}$, arXiv: 1006.2898v1, [math.FA], 15 Jun 2010.
- [5] Krylov, N.V., A generalization of Littlewood-Paley inequality and some other results related to SPDEs, *Ulam Quarterly*, 2 (1994), 16–26.
- [6] Mikulevicius, R. and Pragarauskas, H., Model problem for integro-differential Zakai equation with discontinuous observation processes in Hölder spaces, *Applied Mathematics and Optimization*, 64 (2011), 37–69.
- [7] Mikulevičius, R. and Pragarauskas, H., On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces, *Lithuanian Mathematical Journal* 32-2 (1992), 238–264.
- [8] Mikulevičius, R. and Pragarauskas, H., On Hölder solutions of the integro-differential Zakai equation, *Stochastic Processes and their Applications*, 119 (2009), 3319–3355.
- [9] Mikulevicius, R. and Pragarauskas, H., On L_p - theory for Zakai equation with discontinuous observation process, arXiv:1012.5816v1 [math.PR], 2010.

- [10] Protter, P. E. and Talay, D., The Euler Scheme for Lévy Driven Stochastic Differential Equations, *The Annals of Probability* 25 (1997) 393-423.
- [11] Samorodnitsky, G. and Taqqu, M.S., *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, 1994.
- [12] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1971.
- [13] Stein, E.M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.
- [14] Triebel, H., *Theory of Function Spaces*. Birkhaueser Verlag, 1983.
- [15] Triebel, H., *Theory of Function Spaces II*. Birkhaueser Verlag, 1992.
- [16] Zakai, M., On the optimal filtering of diffusion processes, *Z. Wahrsch.*, **11** (1969), 230-243.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, INSTITUTE OF MATHEMATICS AND INFORMATICS, VILNIUS UNIVERSITY